Lecture 22

Optimal Control of Hybrid Systems

Karl Henrik Johansson

Using dynamic programming, last lecture we derived a sufficient condition for a control policy to be optimal in terms of the Hamilton-Jacobi-Bellman equation. In the current lecture, a necessary condition in the form of the Pontryagin Maximum Principle is presented. It will follow that the optimal control for many continuous-time problems is given by a hybrid control strategy. This is illustrated through a few examples. The lecture ends by stating a result on optimal control of hybrid automata.

1 The Pontryagin Maximum Principle

Let us introduce an optimal control problem similar to the one studied in previous lecture. Consider the continuous-time control system

$$\dot{x}(t) = f(x(t), u(t), t), \qquad x(t_0) = x_0, \tag{1}$$

where $t \in [t_0, t_f]$, $x : [t_0, t_f] \to \mathbb{R}^n$, and $u(t) : [t_0, t_f] \to U$. The control take values in a convex set U. The initial time t_0 is fixed, but the final time t_f is variable. Assume that f is C^1 in x and C^0 in u and t. The cost function is

$$J(u(\cdot)) = \int_{t_0}^{t_f} L(x(s), u(s), s)\, ds + \phi(x(t_f), t_f), \tag{2}$$

where L and ϕ are C^1 in x and C^0 in u and t. The final time t_f is given by the equation

$$\psi(x(t_f), t_f) = 0,$$

where also ψ is C^1 in x and C^0 and t. The optimal control problem is to minimize J with respect to $u(\cdot)$, such that the the dynamics and the final state constraint are fulfilled. The optimal control is denoted $u^* : [t_0, t_f] \to \mathbb{R}$ and the corresponding trajectory $x^* : [t_0, t_f] \to \mathbb{R}^n$. The cost-to-go function is defined as the minimum cost to go from any state $x \in \mathbb{R}^n$ at time $t \in [t_0, t_f]$ to $\{x : \psi(x, t_f) = 0\}$ and is given by

$$J^*(x, t) = \min_{u(\cdot)} \int_{t}^{t_f} L(x(s), u(s), s)\, ds + \phi(x(t_f), t_f).$$

The Hamiltonian is introduced as

$$H(x, p, u, t) = L(x, u, t) + p^T f(x, u, t).$$

The following result gives a necessary condition for a control and a trajectory to be optimal.

Theorem 1 (The Pontryagin Maximum Principle). *If $u^* : [t_0, t_f] \to \mathbb{R}$ and $x^* : [t_0, t_f] \to \mathbb{R}^n$ are optimal control and optimal trajectory, respectively, then*

$$\dot{x}^*(t) = \frac{\partial H^T}{\partial p}(x^*(t), p(t), u^*(t), t) \tag{3}$$

$$\dot{p}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), p(t), u^*(t), t) \tag{4}$$

with boundary conditions

$$\begin{aligned} x^*(t_0) &= x_0 \\ p(t_f) &= \lambda \frac{\partial \psi^T}{\partial x}(x^*(t_f), t_f) + \frac{\partial \phi^T}{\partial x}(x^*(t_f), t_f) \\ H(x^*(t_f), p(t_f), u^*(t_f), t_f) &= -\lambda \frac{\partial \psi}{\partial t}(x^*(t_f), t_f) - \frac{\partial \phi}{\partial t}(x^*(t_f), t_f), \end{aligned}$$

where $\lambda \in \mathbb{R}$. Moreover,

$$u^*(t) = \arg\min_{u \in U} H(x^*(t), p(t), u, t).$$

Proof. Only a sketch for (3) and (4) is given. Equation (3) follows directly from the definition of H. Equation (4) we only derive for the case when J^* is C^2 and u^* is unconstrained, i.e., $u^*(t)$ belongs to the interior of U for all t. Note, however, that the result holds also if these two assumptions are not fulfilled.

From the Hamilton-Jacobi-Bellman equation, it follows that

$$\frac{\partial J^*}{\partial t}(x, t) + \frac{\partial J^*}{\partial x}(x, t) f(x, u^*, t) + L(x, u^*, t) \equiv 0.$$

This is an identity in x and therefore the partial derivative with respect to x is equal to zero, i.e.,

$$\begin{aligned} \frac{\partial^2 J^*}{\partial x \partial t}(x, t) + \frac{\partial^2 J^*}{\partial x^2}(x, t) f(x, u^*, t) + \frac{\partial J^*}{\partial x}(x, t) \frac{\partial f}{\partial x}(x, u^*, t) + \frac{\partial L}{\partial x}(x, u^*, t) \\ + \frac{\partial}{\partial u}\left[\frac{\partial J^*}{\partial x}(x, t) f(x, u^*, t) + L(x, u^*, t)\right] \frac{\partial u^*}{\partial x} = 0. \end{aligned}$$

The last term of the left-hand side is equal to zero due to the assumption that u^* is unconstrained. We may interchange the orders of second partial derivatives since J^* is C^2. This gives for the optimal trajectory x^*,

$$\frac{d}{dt}\left(\frac{\partial J^*}{\partial x}(x^*(t), t)\right) + \frac{\partial J^*}{\partial x}(x^*(t), t) \frac{\partial f}{\partial x}(x^*(t), u^*, t) + \frac{\partial L}{\partial x}(x^*(t), u^*, t) = 0.$$

Introducing

$$p^T(t) = \frac{\partial J^*}{\partial x}(x^*(t), t)$$

and using the definition of H now gives (4).

T The condition for optimality in Theorem 1 does not put any restriction on the smoothness of J^*. This is important in applications, where u^* may be discontinuous and J^* may not be continuously differentiable. Recall that in Theorem 1 in previous lecture it was assumed that J^* is a C^1 function.

The reason why it is called the *maximum* principle and not the *minimum* principle is that in the Russian literature the cost function is usually by convention maximized instead of minimized.

2 Optimal Controllers Modeled as Hybrid system

Applying the Pontryagin Maximum Principle often leads to control policies taking values only on the boundary of U. In these cases, the controller may be implemented as a hybrid automaton. This is illustrated by the following examples.

Example 1 (Time-Optimal Control). Consider the double integrator

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t), \end{aligned} \tag{5}$$

where $u(t) \in [-1, 1]$ and $x(0) = (x_{10}, x_{20})^T$ with $x_{10} > x_{20}$. Let the cost function be

$$J(u(\cdot)) = \int_0^{t_f} dt = t_f$$

and let the constraint on the final state be $\psi(x(t_f)) = x_1(t_f) - x_2(t_f) = 0$. The optimal control problem is thus to minimize the time to reach the line $x_1 = x_2$. The Hamiltonian is equal to

$$H(x, p, u, t) = L(x, u, t) + p^T f(x, u, t) = 1 + p_1 x_2 + p_2 u.$$

Theorem 1 gives

$$\begin{aligned} \dot{p}_1(t) &= 0 \\ \dot{p}_2(t) &= -p_1(t), \end{aligned}$$

with $p_1(t_f) = \lambda$ and $p_2(t_f) = -\lambda$ for some $\lambda \in \mathbb{R}$. Hence, for all $t \in [0, t_f]$,

$$\begin{aligned} p_1(t) &= \lambda \\ p_2(t) &= -\lambda t + \lambda t_f - \lambda. \end{aligned}$$

Moreover,

$$u^*(t) = \arg \min_{u \in [-1,1]} H(x^*(t), p(t), u, t) = \arg \min_{u \in U} [1 + p_1(t) x_2(t) + p_2(t) u] = -\operatorname{sgn} p_2(t).$$

Assuming $t_f > 1$, the optimal control is

$$u^*(t) = \begin{cases} -1, & t < t_f - 1 \\ 1, & t_f - 1 \le t \le t_f. \end{cases}$$

Fig. 1. Fig. 5.1 (p. 245) in [1].

Fig. 2. Hybrid automaton implementing time-optimal control for double integrator.

Using this and integrating (6) backwards from $x_1(t_f) = x_2(t_f) = a$, $a > 1$, gives the switching line $x_1 = 1/2$ and $x_2 < 0$. Figure 1 shows the phase plane together with a few optimal trajectories. The optimal controller is implemented by the hybrid automaton in Figure 4.

Example 2 (Dubin's Vehicle). Consider a model of a vehicle given by the equations

$$\begin{aligned} \dot{x}(t) &= v\cos\theta \\ \dot{y}(t) &= v\sin\theta \\ \dot{\theta}(t) &= u, \end{aligned}$$

where $(x, y) \in \mathbb{R}2$ are the coordinates for the center of the vehicle and $\theta \in [-\pi, \pi]$ is the heading. The control u is the turning rate and is bounded by v/R, where $v > 0$ is the constant forward velocity of the vehicle and $R > 0$ is its lower bound on the turning radius. We are interested in the control problem going from (x_0, y_0, θ_0) to a specified point (x_f, y_f, θ_f) in minimum time. Hence,

$$J(u(\cdot)) = \int_0^{t_f} dt = t_f, \qquad \psi(x, y, \theta) = (x - x_f)^2 + (y - y_f)^2 + (\theta - \theta_f)^2.$$

The Hamiltonian is given by

$$H(x, p, u, t) = L(x, u, t) + p^T f(x, u, t) = 1 + p_1 v\cos\theta + p_2 v\sin\theta + p_3 u,$$

which give the following differential equation for the co-state:

$$\begin{aligned} \dot{p}_1(t) &= 0 \\ \dot{p}_2(t) &= 0 \\ \dot{p}_3(t) &= p_1(t)v\sin\theta(t) - p_2(t)v\cos\theta(t) = p_1(t)\dot{y}(t) - p_2(t)\dot{x}(t). \end{aligned}$$

Fig. 3. Example of an optimal trajectory for Dubin's vehicle.

Fig. 4. Hybrid automaton implementing time-optimal control for double integrator.

Therefore, $p_1(t) = \bar{p}_1$ and $p_2(t) = \bar{p}_2$ are constant and

$$p_3(t) = p_3(0) + \bar{p}_1(y(t) - y_0) + \bar{p}_2(x(t) - y_0).$$

The optimal control is

$$u^*(t) = \arg \min_{|u| \leq v/R} H(x^*(t), p(t), u, t) = \arg \min_{|u| \leq v/R} [up_3(t)] = -\frac{v}{R} \operatorname{sgn} p_3(t).$$

The optimal control consists thus of "turn right" ($p_3 > 0$) and "turn left" ($p_3 < 0$). Allowing relaxed control $u^* = 0$ (corresponding to infinitely fast switching between these two controls), the optimal control also include "go straight." Furthermore, one can show that an optimal path is either of the form $C_a C_b C_c$ or $C_d S_e C_f$, where C_ℓ and S_ℓ indicate circle arc and straight line, respectively, of length ℓ. The lengths fulfill the conditions

$$b \in (\pi R, 2\pi R), \quad a, c \in [0, b], \quad d, f \in [0, 2\pi R), e \in [0, \infty).$$

An example of an optimal trajectory is shown in Figure 3. The optimal controller is implemented by the hybrid automaton in Figure 4. The invariants and reset relations are given by quite complicated expressions and are therefore suppressed in the figure.

Example 3 (Fuller's Phenomenon). Consider the double integrator as in Example 1

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t), \end{aligned} \tag{6}$$

where $u(t) \in [-1, 1]$. Let the cost function now be

$$J(u(\cdot)) = \int_0^{t_f} x_1^2(t)\, dt$$

Fig. 5. Zeno hybrid automaton implementing optimal control for double integrator.

Fig. 6. Hybrid automaton implementing time-optimal control for double integrator.

and let the constraint on the final state be $\psi(x(t_f)) = x_1^2(t_f) + x_2^2(t_f) = 0$. The optimal control is given by a switching curve $x_1 = -ax_2^2 \operatorname{sgn} x_2$, where $a = \ldots$. The hybrid automaton implementing the optimal control is presented in Figure 5. One can show that the hybrid automaton is Zeno, thus given an optimal trajectory having infinitely many switchings in $[0, t_f]$. Figure 6 shows the optimal switching curve together with a trajectory.

3 Optimal Control of Hybrid Automata

A generalization of the Pontryagin Maximum Principle to hybrid automata is presented in this section. To simplify the notation, we consider a hybrid automaton with only two discrete states $\{v_1, v_2\}$ and two edges $\{(v_1, v_2), (v_2, v_1)\}$. To each mode, we associate a differential equation $\dot{x} = f_k(x, u_c)$, where u_c is the continuous control which takes values in a convex set U. The discrete control u_d enters through the invariant and jump conditions as

$$\operatorname{Inv}(v_1) = \{(x, u_d) : \psi_1(x,t) \neq 0 \land u_d = 1\}$$

and

$$\operatorname{Jump}(v_1, x) = \begin{cases} (v_2, x), & \text{if } \psi_1(x,t) = 0 \lor u_d = 2 \\ \emptyset, & \text{otherwise,} \end{cases}$$

and similar for v_2. The described hybrid automaton is an open hybrid automaton (see Lecture 5) with no outputs, i.e., $H = (V, X, U, f, \text{Init}, \text{Inv}, \text{Jump})$.

An execution of H is a four-tuple $\chi = (\tau, v, x, u)$, where τ is a finite time trajectory, v the discrete evolution, x the continuous evolution, and $u = (u_c, u_d)$ the control. For the time trajectory, we assume that $\tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N$. We call the transition at time $\tau_i' = \tau_{i+1}$ an autonomous transition if $\psi(x(\tau_i'), \tau_i') = 0$ and a controlled transition if $u_d(\tau_i') \neq u_d(\tau_{i+1})$. We assume that an autonomous and a controlled transition cannot take place at the same time. Let the cost function be

$$J(\chi) = \sum_{i=0}^{N} \int_{\tau_i}^{\tau_i'} L_{v(t)}(x(t), u_c(t))\, dt.$$

An execution $\chi^* = (\tau^*, v^*, x^*, u^*)$ is called optimal, if $J(\chi^*) \leq J(\chi)$ for all $\chi = (\tau, v, x, u)$ with $\tau = \tau^*$ and $(v_0, x_0) = (v_0^*, x_0^*)$. For $k = 1, 2$, define the Hamiltonian

$$H_k(x, p, u_c) = L_k(x, u_c) + p^T f_k(x, u_c).$$

Then, we have the following generalization of the Pontryagin Maximum Principle.

Theorem 2. *If χ^* is optimal, then there exists $p : \tau \to \mathbb{R}^n$ such that*

– *for all $t \in (\tau_i, \tau_i')$, and*

$$\begin{aligned}
\dot{x}^*(t) &= \frac{\partial H^T_{v^*(t)}}{\partial p}(x^*(t), p(t), u^*(t)) \\
\dot{p}(t) &= -\frac{\partial H^T_{v^*(t)}}{\partial x}(x^*(t), p(t), u^*(t)) \\
H_{v^*(t)}(x^*(t), p(t), u_c^*(t)) &= \min\{\inf_{w \in U} H_1(x^*(t), p(t), w), \inf_{w \in U} H_2(x^*(t), p(t), w)\},
\end{aligned}$$

– *for all autonomous transitions at $\tau_i' = \tau_{i+1}$,*

$$\begin{aligned}
p(\tau_{i+1}) &= p(\tau_i') - \lambda \frac{\partial \psi^T_{v^*(\tau_i')}}{\partial x}(x^*(\tau_i')) \\
H_{v^*(\tau_{i+1})}(x^*(\tau_{i+1}), p(\tau_{i+1}), u_c^*(\tau_{i+1})) &= H_{v^*(\tau_i')}(x^*(\tau_i'), p(\tau_i'), u_c^*(\tau_i')) \\
&\quad + \lambda \frac{\partial \psi^T_{v^*(\tau_i')}}{\partial t}(x^*(\tau_i')),
\end{aligned}$$

– *for all controlled transitions at $\tau_i' = \tau_{i+1}$,*

$$\begin{aligned}
p(\tau_{i+1}) &= p(\tau_i') \\
H_{v^*(\tau_{i+1})}(x^*(\tau_{i+1}), p(\tau_{i+1}), u_c^*(\tau_{i+1})) &= H_{v^*(\tau_i')}(x^*(\tau_i'), p(\tau_i'), u_c^*(\tau_i')).
\end{aligned}$$

Example 4. To be included.

Background

There are many textbooks on optimal control, e.g., [10, 2, 4, 6, 8, 14, 1]. Example 1 is taken from [1]. Example 2 is based on [12]. Fuller's phenomenon (Example 3) is extensively discussed in [9]. Section 3 is derived from [11], where particularly the linear quadratic control for hybrid systems is studied. See [3, 5, 13] for other hybrid optimal control problem. Computational issues are discussed in [7].

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