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# Model predictive control for max-min-plus-scaling systems

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**Abstract**

We further extend the model predictive control framework, which is very popular in the process industry due to its ability to handle constraints on inputs and outputs, to a class of discrete event systems that can be modeled using the operations maximization, minimization, addition and scalar multiplication. This class encompasses max-plus-linear systems, min-max-plus systems, bilinear max-plus systems and polynomial max-plus systems. In general the model predictive control problem for max-min-plus-scaling systems leads to a nonlinear non-convex optimization problem, that can also be reformulated as an optimization problem over the solution set of an extended linear complementarity problem. We also show that under certain conditions the optimization problem reduces to a convex programming problem, which can be solved very efficiently.

## 1 Introduction

Model predictive control (MPC) is a very popular controller design method in the process industry. MPC provides many attractive features: it is an easy-to-tune method, it is applicable to multivariable systems, it can handle constraints in a systematic way, and it is capable of tracking pre-scheduled reference signals.

Usually MPC uses discrete-time models. We will extend MPC to a class of discrete event systems that can be modeled using maximization, minimization, addition and scalar multiplication, and that are called max-min-plus-scaling (MMPS) systems. Typical examples of MMPS systems are digital circuits, computer networks, telecommunication networks, and manufacturing plants. A key advantage of conventional MPC is that it allows the inclusion of constraints on the inputs and outputs. This is also one of the main reasons why we introduce MPC for MMPS systems. Furthermore, MPC uses a receding horizon strategy which allows us to regularly update the model of the system.

The work presented in this paper unifies our previous results on MPC for max-plus-linear systems [11], max-min-plus systems [10], and first-order linear hybrid systems subject to saturation [8], since all these classes of systems are in fact special subclasses of MMPS systems. In addition, in [15] it has been shown that the class of MMPS systems coincides with the class of mixed-logic dynamic (MLD) systems [3], which includes piecewise affine dynamic systems, linear hybrid systems, finite state machines, linear systems, linear systems with discrete inputs, bilinear systems with discrete inputs, etc. Therefore, MMPS form a interesting and relevant subclass of hybrid systems.

In [3] Bemporad and Morari have developed an MPC method for MLD systems. The main difference between MPC for MLD systems and MPC for MMPS systems is that MLD-MPC requires the solution of mixed integer-real quadratic optimization problems whereas in the MMPS-MPC optimization problems all variables are real-valued. However, in general both MLD-MPC and MMPS-MPC lead to computationally hard optimization problems. Therefore, we also investigate under which conditions MMPS-MPC leads to convex optimization problems.

## 2 Max-min-plus-scaling systems

We use the symbol $\vee$ to denote maximization and $\wedge$ to denote minimization. So if $a,b \in \mathbb{R}$ then $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

**Definition 2.1** *A max-min-plus-scaling (MMPS) expression $f$ of the variables $x_1,\ldots,x_n$ is defined by the grammar*[1]

$$f := x_i \,|\, \alpha \,|\, f_k \vee f_l \,|\, f_k \wedge f_l \,|\, f_k + f_l \,|\, \beta f_k$$

*with $i \in \{1,\ldots,n\}$, $\alpha,\beta \in \mathbb{R}$, and where $f_k$ and $f_l$ are again MMPS expressions.*

Some examples of MMPS expressions of the variables $x_1$, $x_2$, $x_3$ in conventional notation are $x_1 - x_2 + 3$, $\max(\min(3x_1,-x_2),x_1+x_3)$, and $x_1 - 2\max(x_1+3x_2-4x_3, x_1-\min(x_1,x_2-x_3,\max(x_3,x_1-x_2-x_3)))$.

Now we consider discrete event systems that can be described by state space equations of the following form:

$$x(k) = \mathscr{M}_x(x(k-1),u(k)) \tag{1}$$

$$y(k) = \mathscr{M}_y(x(k),u(k)) \tag{2}$$

[1] The symbol | stands for "or". Also note that the definition is recursive. So an MMPS expression is a variable or a constant, or the maximum or minimum or sum of two MMPS expressions, or a scalar multiple of an MMPS expression.

where $\mathscr{M}_x$ and $\mathscr{M}_y$ are MMPS expressions. In general these expressions may even depend on the event counter $k$. For a discrete event system $x(k)$ would typically contain the time instants at which the internal events occur for the $k$th time, $u(k)$ the time instants at which the input events occur for the $k$th time, and $y(k)$ the time instants at which the output events occur for the $k$th time.

The model (1)–(2) can be considered as a generalized framework that encompasses several special subclasses of discrete event systems such as max-plus-linear discrete event systems [2, 7], max-min-plus systems [14, 16, 17], max-plus-bilinear[2] systems, and max-plus-polynomial[3] systems.

Note that the class of MMPS systems is a non-trivial superset of the max-min-plus systems considered in [14, 16] since in contrast to the min-max expressions of [14, 16] we also allow addition of two MMPS expressions and scaling in the definition of MMPS expressions. Furthermore, the min-max-plus systems considered in [14, 16] are autonomous (i.e., only the state is considered, and there is no input or explicit output), whereas in the model (1)–(2) we have included inputs and outputs.

## 3 Model predictive control

In this section we give a short introduction to MPC for deterministic nonlinear discrete-time systems. Since we will only consider the deterministic, i.e. noiseless, case for MMPS systems, we will also omit the noise terms in this introduction to MPC. More extensive information on MPC for (linear and nonlinear) discrete-time systems can be found in [1, 4, 5, 6, 13] and the references therein.

Consider a plant with $m$ inputs and $l$ outputs that can be modeled by a nonlinear discrete-time state space description of the following form:

$$x(k) = f(x(k-1), u(k)) \tag{3}$$

$$y(k) = h(x(k), u(k)) \tag{4}$$

where $f$ and $h$ are smooth functions of $x$ and $u$.

**Remark** Apart from the fact that in (1)–(2) the components of the input, the output and the state are event times, an important difference between the descriptions (1)–(2) and (3)–(4) is that the counter $k$ in (1)–(2) is an event counter (and event occurrence instants are in general not equidistant), whereas in (3)–(4) $k$ is a sample counter that increases each clock cycle.

In MPC we consider the future evolution of the system over a given prediction horizon $N_\mathrm{p}$. For the system (3)–(4) we can make an estimate $\hat{y}(k+j|k)$ for the output at sample step $k+j$ based on the state at step $k$ and the future inputs $u(k+i)$, $i = 0, 1, \ldots, j$. Using successive substitution, we obtain an expression of the following form:

$$\hat{y}(k+j|k) = F_j(x(k-1), u(k), \ldots, u(k+j))$$

for $j = 0, 1, \ldots, N_\mathrm{p} - 1$. If we define the vectors

$$\tilde{u}(k) = \begin{bmatrix} u^T(k) & \ldots & u^T(k+N_\mathrm{p}-1) \end{bmatrix}^T$$
$$\tilde{y}(k) = \begin{bmatrix} \hat{y}^T(k|k) & \ldots & \hat{y}^T(k+N_\mathrm{p}-1|k) \end{bmatrix}^T \quad ,$$

we can derive the expression

$$\tilde{y}(k) = \tilde{F}(x(k-1), \tilde{u}(k)) \quad ,$$

which characterizes the estimated future evolution of the output of the system at sample step $k$ over the prediction horizon $N_\mathrm{p}$ for the input sequence $u(k), u(k+1), \ldots, u(k+N_\mathrm{p}-1)$.

The cost criterion $J$ used in MPC reflects the reference tracking error ($J_\mathrm{out}$) and the control effort ($J_\mathrm{in}$):

$$\begin{aligned} J(k) &= J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k) \\ &= \big(\tilde{y}(k) - \tilde{r}(k)\big)^T \big(\tilde{y}(k) - \tilde{r}(k)\big) + \lambda \tilde{u}^T(k)\tilde{u}(k) \end{aligned}$$

where $\lambda$ is a nonnegative integer, and $\tilde{r}(k)$ contains the reference signal (defined similarly to $\tilde{y}(k)$). In practical situations, there will be constraints on the input and output signals (caused by limited capacity of buffers, limited transportation rates, saturation, etc.) This is reflected in the nonlinear constraint function

$$C_\mathrm{c}(k, \tilde{u}(k), \tilde{y}(k)) \leqslant 0 \quad .$$

The MPC problem at sample step $k$ consists in minimizing $J(k)$ over all possible future input sequences subject to the constraints. This is usually a non-convex optimization problem. To reduce the complexity of the optimization problem a control horizon $N_\mathrm{c}$ is introduced in MPC, which means that the input is taken to be constant beyond sample step $k+N_\mathrm{c}$:

$$u(k+j) = u(k+N_\mathrm{c}-1) \;\text{ for } j = N_\mathrm{c}, \ldots, N_\mathrm{p}-1.$$

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $u(k), u(k+1), \ldots, u(k+N_\mathrm{c}-1)$, only the first element of the optimal sequence ($u(k)$) is applied to the system. Next the horizon is shifted and a new MPC optimization is performed for sample step $k+1$.

So the MPC problem at sample step $k$ for the nonlinear discrete-time system described by (3)–(4) is defined as follows:

[2] Max-plus-bilinear systems are an extension of max-plus-linear systems where we also allow max-plus-algebraic cross-products between a state component and an input component in the right-hand side of the state equation and the output equation.

[3] Max-plus-polynomial systems are an extension of max-plus-linear systems where we also allow max-plus-algebraic polynomial expressions of the state components and the input components in the right-hand side of the state equation and the output equation. Note that models with a right-hand side of the form $\max(A_1 x(k), A_2 x(k), \ldots)$, which can be used in the design of traffi c signal switching schemes [18], also belong to this class.

Find the input sequence $\{u(k),\ldots,u(k+N_\mathrm{p}-1)\}$ that minimizes the cost criterion $J(k)$ subject to the evolution equations (3)–(4) of the system, the nonlinear constraint $C_\mathrm{c}(k,\tilde{u}(k),\tilde{y}(k)) \leqslant 0$ and the control horizon constraint $u(k+j)=u(k+N_\mathrm{c}-1)$ for $j=N_\mathrm{c},\ldots,N_\mathrm{p}-1$.

Recall that due to the receding horizon approach this problem has to be solved at each sample step $k$. Note however that the use of a control horizon leads to a reduction of the number of optimization variables. This results in a decrease of the computational burden, a smoother controller signal (because of the emphasis on the average behavior rather than on aggressive noise reduction), and a stabilizing effect.

## 4 Model predictive control for MMPS systems

In [11] we have extended and adapted the MPC framework from linear discrete-time systems to max-plus-linear discrete event systems, while using — as for as possible — analogous constraints and cost criteria for both types of systems. Since the constraints and objective functions introduced there can also be used for MMPS discrete event systems, we will not discuss them extensively here but only repeat the most important conclusions and results here.

Just as in MPC for nonlinear discrete-time systems, we also define the MPC cost criterion for MMPS systems as $J(k)=J_\mathrm{out}(k)+\lambda J_\mathrm{in}(k)$, where $J_\mathrm{out}(k)$ is related to the output and $J_\mathrm{in}(k)$ is related to the input. For $J_\mathrm{out}(k)$ we could e.g. take the tardiness

$$J_{\mathrm{out},1}(k)=\sum_{i=1}^{lN_\mathrm{p}} \max(\tilde{y}_i(k)-\tilde{r}_i(k),0) \quad .$$

which penalizes the delays w.r.t. the due dates $\tilde{r}$. For perishable goods, where we want to minimize the differences between the due dates and the actual output time instants, we could take

$$J_{\mathrm{out},2}(k)=\sum_{i=1}^{lN_\mathrm{p}} |\tilde{y}_i(k)-\tilde{r}_i(k)| \quad .$$

Using the conventional MPC cost criterion $\tilde{u}^T(k)\tilde{u}(k)$ for $J_\mathrm{in}$ would lead to a minimization of the input time instants, which could result in internal buffer overflows. Therefore, a better objective is to *maximize* the input time instants:

$$J_{\mathrm{in},0}(k)=-\tilde{u}^T(k)\tilde{u}(k)$$

or

$$J_{\mathrm{in},1}(k)=-\sum_{i=1}^{mN_\mathrm{p}} \tilde{u}_i(k).$$

Of course several other choices are possible for $J_\mathrm{out}(k)$ and $J_\mathrm{in}(k)$ (including the one used in [3] for MLD systems).

In the context of discrete event systems typical constraints are

$$\begin{aligned} &a_1(k+j) \leqslant \Delta u(k+j-1) \leqslant b_1(k+j) && \text{for } j=1,\ldots,N_\mathrm{c} \\ &a_2(k+j) \leqslant \Delta \hat{y}(k+j|k) \leqslant b_2(k+j) && \text{for } j=1,\ldots,N_\mathrm{p} \\ &\hat{y}(k+j|k) \leqslant r(k+j) && \text{for } j=1,\ldots,N_\mathrm{p}\,, \end{aligned}$$

where $\Delta u(k+j)=u(k+j)-u(k+j-1)$. Note that all these constraints can be rewritten as a linear constraint of the form

$$A_\mathrm{c}(k)\,\tilde{u}(k)+B_\mathrm{c}(k)\,\tilde{y}(k) \leqslant c_\mathrm{c}(k) \quad .$$

In general we can consider constraints of the form

$$\mathscr{M}_\mathrm{c}(k,\tilde{u}(k),\tilde{y}(k),x(k-1)) \leqslant 0$$

where $\mathscr{M}_\mathrm{c}$ is an MMPS expression.

Since for the type hybrid systems we are considering the input sequence usually corresponds to occurrence times of consecutive events, it should always be nondecreasing. Therefore, we also add the condition

$$\Delta u(k+j) \geqslant 0 \qquad \text{for } j=0,\ldots,N_\mathrm{p}-1\,.$$

Finally, we introduce a control horizon condition in order to reduce the number of variables in the optimization problem: the input "rate" should stay constant beyond step $k+N_\mathrm{c}$, i.e. $\Delta u(k+j)=\Delta u(k+N_\mathrm{c}-1)$ for $j=N_\mathrm{c},\ldots,N_\mathrm{p}-1$, or equivalently

$$\Delta^2 u(k+j)=0 \quad \text{for } j=N_\mathrm{c},\ldots,N_\mathrm{p}-1\,.$$

So we finally obtain the following problem at event step $k$:

Find the input sequence vector $\tilde{u}(k)$ that minimizes the cost criterion $J(k)$ subject to

$$\hat{x}(k+j|k)=\mathscr{M}_x(\hat{x}(k+j-1|k),u(k+j)) \quad \text{for } j=0,\ldots,N_\mathrm{p}-1, \tag{5}$$

$$\hat{y}(k+j|k)=\mathscr{M}_y(\hat{x}(k+j-1|k),u(k+j)) \quad \text{for } j=0,\ldots,N_\mathrm{p}-1, \tag{6}$$

$$\mathscr{M}_\mathrm{c}(k,\tilde{u}(k),\tilde{y}(k),x(k-1)) \leqslant 0 \tag{7}$$

$$\Delta u(k+j) \geqslant 0 \qquad \text{for } j=0,\ldots,N_\mathrm{p}-1, \tag{8}$$

$$\Delta^2 u(k+j)=0 \qquad \text{for } j=N_\mathrm{c},\ldots,N_\mathrm{p}-1, \tag{9}$$

with $\hat{x}(k-1|k)=x(k-1)$.

This problem will be called the MMPS-MPC problem for event step $k$. Note that in this case we also use a receding horizon approach in which in each step we effectively apply only the first input sample.

## 5 Algorithms to solve the MMPS-MPC problem

### 5.1 The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) is defined as follows [9]:

Given $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{q\times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and $m$ subsets $\phi_1,\ldots,\phi_m$ of $\{1,\ldots,p\}$, find $z \in \mathbb{R}^n$ such that

$$\prod_{i\in\phi_j}(Az-c)_i = 0 \quad \text{for } j = 1,\ldots,m, \tag{10}$$

subject to $Az \geqslant c$ and $Bz = d$, or show that no such $z$ exists.

Equation (10) represents the complementarity condition of the ELCP. One possible interpretation of this condition is the following: each set $\phi_j$ corresponds to a group of inequalities of $Az \geqslant c$ and in each group at least one inequality should hold with equality, i.e. the corresponding residue should be equal to 0. So for each $j$ there should exist an index $i \in \phi_j$ such that $(Az-c)_i = 0$.

In general, the solution set of the ELCP defined above consists of the union of faces of the polyhedron defined by the system of linear equations and inequalities ($Az \geqslant c$ and $Bz = d$) of the ELCP. In [9] we have developed an algorithm to compute the complete solution set of an ELCP. This algorithm yields a description of the solution set by vertices, extreme rays and a basis of the linear subspace corresponding to the largest affine subspace of the solution set.

### 5.2 Link between the MMPS-algebraic MPC problem and the ELCP

Let us now show how the MMPS-MPC problem can be reformulated using the ELCP. This will be done by showing that each of the 6 constructions for MMPS expressions fit the ELCP framework:

- Expressions of the form $f = x_i$, $f = \alpha$, $f = f_k + f_l$ and $f = \beta f_k$ (or their combinations) result in linear equations of the form $Bz = d$ where $z$ contains the variables[4] $f$, $x_i$, $f_k$ and $f_l$.
- An expression of the form $f = f_k \vee f_l = \max(f_k, f_l)$ can be rewritten as

$$\begin{aligned} &f \geqslant f_k \\ &f \geqslant f_l \\ &f = f_k \text{ or } f = f_l \end{aligned}$$

  or equivalently

$$\begin{aligned} &f - f_k \geqslant 0 \\ &f - f_l \geqslant 0 \\ &(f - f_k)(f - f_l) = 0 \ , \end{aligned}$$

  which is an ELCP.
- In a similar way an expression of the form $f = f_k \wedge f_l = \min(f_k, f_l)$ can be rewritten as

$$\begin{aligned} &f_k - f \geqslant 0 \\ &f_l - f \geqslant 0 \\ &(f_k - f)(f_l - f) = 0 \ , \end{aligned}$$

  which is also an ELCP.

[4] In this case $f$, $f_k$ and $f_l$ are dummy variables.

This implies that by introducing additional dummy variables if necessary, any MMPS expression can be recast as an ELCP. Furthermore, two or more ELCPs can be combined into one large ELCP. The constraints (8)–(9) just yield additional linear (in)equalities. So the system (5)–(9), which defines the feasible set of the MMPS-MPC problem, can be rewritten as an ELCP. We can compute a compact parametric description of the solution set of an ELCP using the algorithm of [9]. In order to determine the optimal MPC policy we then have to determine for which values of the parameters the objective function $J(k)$ over the solution set of the ELCP that corresponds to (5)–(9). The algorithm of [9] to compute the solution set of a general ELCP requires exponential execution times. This implies that the ELCP approach sketched above is not feasible if $N_c$, $m$ or $l$ are large. In that case we could use standard multi-start nonlinear non-convex local optimization methods to compute the optimal control policy. However, in the next section we will show that under certain conditions the MMPS-MPC problem leads to a convex optimization problem, which can be solved very efficiently.

### 5.3 Monotonic objective functions and constraints

**Definition 5.1**

*A max-plus-positive-scaling (MaxPPS) expression $f$ of the variables $x_1,\ldots,x_n$ is defined by the grammar*

$$f := \alpha x_i \,|\, \beta \,|\, f_k \vee f_l \,|\, f_k + f_l \,|\, \rho f_k$$

*with $i \in \{1,\ldots,n\}$, $\alpha,\beta,\rho \in \mathbb{R}$, $\rho > 0$ and where $f_k$ and $f_l$ are again MaxPPS expressions.*
*A min-plus-positive-scaling (MinPPS) expression $f$ of the variables $x_1,\ldots,x_n$ is defined by the grammar*

$$f := \alpha x_i \,|\, \beta \,|\, f_k \wedge f_l \,|\, f_k + f_l \,|\, \rho f_k$$

*with $i \in \{1,\ldots,n\}$, $\alpha,\beta,\rho \in \mathbb{R}$, $\rho > 0$ and where $f_k$ and $f_l$ are again MinPPS expressions.*

In the remainder of this section we consider the MPC problem for a subclass of MMPS systems that can be described by the following state space model:

$$x^\vee(k) = \mathscr{M}_x^\vee(x^\vee(k-1), u(k)) \tag{11}$$
$$x^\wedge(k) = \mathscr{M}_x^\wedge(x^\wedge(k-1), u(k)) \tag{12}$$
$$y^\vee(k) = \mathscr{M}_y^\vee(x^\vee(k), u(k)) \tag{13}$$
$$y^\wedge(k) = \mathscr{M}_y^\wedge(x^\wedge(k), u(k)) \tag{14}$$

where $\mathscr{M}_x^\vee$ and $\mathscr{M}_y^\vee$ are MaxPPS expressions, and $\mathscr{M}_x^\wedge$ and $\mathscr{M}_y^\wedge$ are MinPPS expressions. The vector $x(k) = [\,(x^\vee)^T(k)\ \ (x^\wedge)^T(k)\,]^T$ is the state of the system at event step $k$, and $y(k) = [\,(y^\vee)^T(k)\ \ (y^\wedge)^T(k)\,]^T$ is the output of the system at event step $k$. Furthermore, we consider a linear constraint instead of the general MMPS constraint (7). So the

MPC constraints are

$$A_c(k)\,\tilde{u}(k) + B_c(k)\,\tilde{y}(k) \leqslant c_c(k) \tag{15}$$

$$\Delta u(k+j) \geqslant 0 \qquad \text{for } j = 0,\ldots,N_p - 1, \tag{16}$$

$$\Delta^2 u(k+j) = 0 \qquad \text{for } j = N_c,\ldots,N_p - 1. \tag{17}$$

Now we consider the *relaxed MMPS-MPC problem* for the system described by (11)–(14). This problem is defined by the evolution equations (11)–(14) and the constraints (15)–(17) but with the =-sign in (11) and (13) replaced by a $\geqslant$-sign, and the =-sign in (12) and (14) replaced by a $\leqslant$-sign:

$$x^{\vee}(k) \geqslant \mathscr{M}_x^{\vee}(x^{\vee}(k-1), u(k)) \tag{18}$$

$$x^{\wedge}(k) \leqslant \mathscr{M}_x^{\wedge}(x^{\wedge}(k-1), u(k)) \tag{19}$$

$$y^{\vee}(k) \geqslant \mathscr{M}_y^{\vee}(x^{\vee}(k), u(k)) \tag{20}$$

$$y^{\wedge}(k) \leqslant \mathscr{M}_y^{\wedge}(x^{\wedge}(k), u(k)) \ . \tag{21}$$

These equations describe a convex set[5]. Furthermore, the constraints (15)–(17) are linear and thus convex. As a consequence, the set of feasible solutions of the relaxed MMPS-MPC problem is convex. Hence, the relaxed problem is much easier to solve numerically.

We say that a function $F$ is a monotonically nondecreasing (nonincreasing) function of $y$ if $y^* \leqslant y^\sharp$ implies that $F(y^*) \leqslant F(y^\sharp)$ $\big(F(y^*) \geqslant F(y^\sharp)\big)$. Using a reasoning that is an extension of that used in [11] for the max-plus-algebraic MPC, it can be shown that if the objective function $J(k)$ and the linear constraints are monotonically nondecreasing as function of $\tilde{y}^{\vee}(k)$ and monotonically nonincreasing functions of $\tilde{y}^{\wedge}(k)$, then the optimal solution of the relaxed MMPS-MPC problem can be transformed into a solution of the original MPC problem:

**Theorem 5.2** *Consider an MMPS system that can be modeled by (11)–(14). Let the objective function $J(k)$ and the mapping $\tilde{y}(k) \to B_c(k)\tilde{y}(k)$ be monotonically nondecreasing functions of $\tilde{y}^{\vee}(k)$ (and $x^{\vee}(k)$) and monotonically nonincreasing functions of $\tilde{y}^{\wedge}(k)$ (and $x^{\wedge}(k)$), Let $(\tilde{u}^*(k), \tilde{y}^*(k))$ be an optimal solution of the relaxed MMPS-MPC problem. If we define $\tilde{y}^\sharp(k)$ by*

$$x^{\vee,\sharp}(k+j|k) = \mathscr{M}_x^{\vee}\big(x^{\vee,\sharp}(k+j-1|k), u^*(k+j)\big)$$

$$x^{\wedge,\sharp}(k+j|k) = \mathscr{M}_x^{\wedge}\big(x^{\wedge,\sharp}(k+j-1|k), u^*(k+j)\big)$$

$$y^{\vee,\sharp}(k+j|k) = \mathscr{M}_y^{\vee}\big(x^{\vee,\sharp}(k+j|k), u^*(k+j)\big)$$

$$y^{\wedge,\sharp}(k+j|k) = \mathscr{M}_y^{\wedge}\big(x^{\wedge,\sharp}(k+j|k), u^*(k+j)\big)$$

*for $j = 0,1,\ldots,N_p - 1$ and with $x^{\vee,\sharp}(k-1|k) = x^{\vee}(k-1)$ and $x^{\wedge,\sharp}(k-1|k) = x^{\wedge}(k-1)$, then $(\tilde{u}^*(k), \tilde{y}^\sharp(k))$ is an optimal solution of the original MMPS-MPC problem.*

[5] It is easy to verify that any MaxPPS expression $f$ can be written in a *conjunctive normal form* $f = f_1 \vee f_2 \vee \cdots \vee f_n$ where $f_1,\ldots,f_n$ are affine expressions. Hence, e.g., the $i$th subequation of (18) can be rewritten as $x_i^{\vee}(k+1) \geqslant f_1 \vee f_2 \vee \cdots \vee f_n$ where $f_1, f_2,\ldots,f_n$ are affine expressions. This inequality is equivalent to the system of linear inequalities $x_i^{\vee}(k+1) \geqslant f_1$, $x_i^{\vee}(k+1) \geqslant f_2, \ldots x_i^{\vee}(k+1) \geqslant f_n$, which defines a convex set.

**Proof:** The proof of this theorem is an extension to MMPS systems of the proof of Theorem 5.1 of [11] for the max-plus-algebraic MPC. In fact, the proof is similar to the proof of the property that a feasible linear programming problem with a finite optimal solution always has an optimal solution in which at least one of the constraints is active. For more details the interested reader is referred to [12]. ■

Note that we can always obtain an objective function that is a monotonically nondecreasing function of $\tilde{y}^{\vee}(k)$ and a monotonically nonincreasing function of $\tilde{y}^{\wedge}(k)$ by eliminating $\tilde{y}(k)$ from the expression for $J(k)$ using the evolution equations (11)–(14) before relaxing the problem. However, some of the properties (such as convexity or linearity) of the original objective function may be lost in that way.

Recall that the relaxed MMPS-MPC problem has a convex feasible set. So if Theorem 5.2 applies the optimal MPC policy can be computed much more efficiently than in the general case. If in addition the objective function is convex (e.g. if $J(k)$ equals $-J_{\text{in},0}(k)$, $\pm J_{\text{in},1}(k)$, $J_{\text{out},1}(k)$ or a weighted combination of these objective functions), we finally get a convex optimization problem, which can be solved efficiently using, e.g. an interior point method. Since $J_{\text{in},1}(k)$ is a linear function, the problem even reduces to a linear programming problem for $J(k) = \pm J_{\text{in},1}(k)$, which can be solved very efficiently. Furthermore, it easy to verify that for $J(k) = J_{\text{out},1}(k)$ the problem can also be reduced to a linear programming problem by introducing some additional dummy variables.

## 6 Discussion

We have further extended the popular MPC framework from nonlinear discrete-time systems to MMPS discrete event systems. The reason for using an MPC approach for MMPS systems is the same as for conventional linear systems: MPC allows the inclusion of constraints on the inputs and outputs, it is an easy-to-tune method, and it is flexible for structure changes (since the optimal strategy is recomputed every time step or event step so that model changes can be taken into account as soon as they are identified).

We have also presented some methods to solve the MMPS-MPC problem. In general this leads to a nonlinear nonconvex optimization problem. If the state and output equations can be split in max-plus-scaling and min-plus-scaling parts that are decoupled, and if the objective function and the constraints are monotonic functions of the states and the outputs, then we can relax the MMPS-MPC problem to problem with a convex set of feasible solutions. If in addition the objective function is convex or linear, this leads to a convex or linear optimization problem, which can be solved very efficiently.

An important topic for further research is the investigation of the effects of the three tuning parameters (the input cost weight $\lambda$, the prediction horizon $N_p$ and the control horizon $N_c$) and the selection of appropriate values for these tuning parameters.

Another topic for further investigation is the comparison of the MMPS-MPC approach with the MPC approach for mixed-logic dynamic (MLD) systems introduced in [3] since MMPS systems are equivalent to MLD systems [15]. The main difference between MPC for MLD systems and MPC for MMPS systems is that MLD-MPC requires the solution of *mixed integer-real* quadratic optimization problems whereas MMPS-MPC requires the solution of optimization problems with *real-valued* variables. In general both MLD-MPC and MMPS-MPC result in computationally hard optimization problems. Therefore, we will also compare the performance of MLD-MPC and MMPS-MPC for several special subclasses of MMPS or MLD systems. Moreover, it is also an open question whether there exist other subclasses of MMPS and MLD systems (apart from the decoupled max-plus-scaling/min-plus-scaling systems considered in this paper) for which the resulting MPC optimization problem can be recast as a problem that can be solved efficiently.

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